Intertwining techniques for actions of C*-tensor categories

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Outline

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Roadmap to classification

A,B "well-behaved" C*-algebras (plus extra structure) when is $A \cong B$?

Roadmap to classification

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Theorem (Elliott 76')

Unital AF C-algebras A, B are isomorphic if and only if* $(K_0(A), K_0(A)^+, [1_A]) \cong (K_0(B), K_0(B)^+, [1_B])$.

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Theorem (Kirchberg–Phillips 00')

If A and B unital, separable, amenable, simple purely infinite C-algebras satisfying the UCT then $A \cong B$ if and only if $((K_0(A), [1_A]), K_1(A)) \cong ((K_0(B), [1_B]), K_1(B))$*

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Roadmap to classification

What is the structure of proof? Let

$$\mathrm{Inv} : \textbf{C*alg} \to \mathcal{K}$$

be a suitable functor to be used for classification.

Roadmap to classification

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Uniqueness: Show that since $\mathrm{Inv}(\psi \circ \phi) = \mathrm{Inv}(\mathrm{id}_A)$ and $\mathrm{Inv}(\phi \circ \psi) = \mathrm{Inv}(\mathrm{id}_B)$ then $\phi \circ \psi \approx_{a.u} \mathrm{id}_B$ or $\psi \circ \phi \approx_{a.u} \mathrm{id}_A$.

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Definition

$\phi, \psi : A \to B$ are approximately unitary equivalent ($\phi \approx_{a.u} \psi$) if there exists $u_n \in U(M(B))$ such that

$$u_n \phi(a) u_n^* \underset{n \to \infty}{\longrightarrow} \psi(a), \quad \forall a \in A.$$

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Intertwining: As ψ and ϕ are mutually inverse to one another up to approximate unitary equivalence, you may tweak them to make them genuinely inverse to each other.

Roadmap to classification

Two sided intertwining Theorem

If $\phi : A \to B$ and $\psi : B \to A$ satisfy $\psi \circ \phi \approx_{a.u} \mathrm{id}_A$ and $\phi \circ \psi \approx_{a.u} \mathrm{id}_B$ then there exists $\Phi \approx_{a.u} \phi$ and $\Psi \approx_{a.u} \psi$ such that $\Psi \circ \Phi = \mathrm{id}_A$ and $\Phi \circ \Psi = \mathrm{id}_B$.

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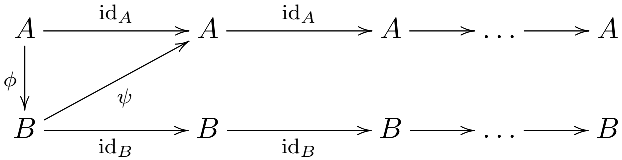
The quotient functor **C*alg** $\to$ **C*alg**$/ \approx_{a.u}$ is full on isomorphisms

Sketch proof

Theorem

The quotient functor **C*alg** $\to$ **C*alg**$/ \approx_{a.u}$ *is full on isomorphisms*



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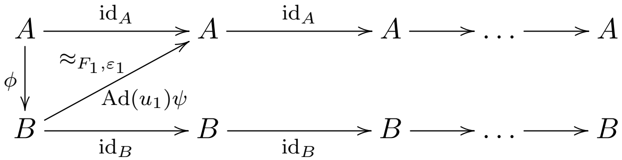

$\phi \circ \mathrm{Ad}(u_1)\psi = \mathrm{Ad}(\phi(u_1))\phi\psi \approx_{a.u} \mathrm{id}_B,$ "extendible"

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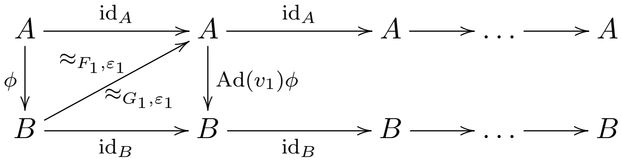

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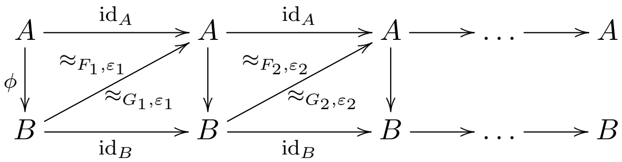

choosing ε_i summable, $\overline{\cup F_i} = A$ and $\overline{\cup G_i} = B$ the construction yields that

$$\lim_{n\to\infty} \mathrm{Ad}(v_n)\phi$$

$$\lim_{n\to\infty} \mathrm{Ad}(u_n)\psi$$

exist and are mutually inverse. $\square$

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Theorem

The quotient functor **C*alg** $\to$ **C*alg**$/ \approx_{a.u}$ *is full on isomorphisms*

We required:

- A well-behaved notion of unitary equivalence.
- A complete, metrisable topology at the level of morphisms.

"Towards a theory of classification"-Elliott 10'

Classifying Γ-C*-algebras

Theorem (Kirchberg–Phillips 00')

If A and B are unital Kirchberg algebras satisfying the UCT then $A \cong B$ iff $((K_0(A), [1_A]), K_1(A)) \cong ((K_0(B), [1_B]), K_1(B))$.

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Theorem (Gabe–Szabó 22')

If $\alpha : \Gamma \curvearrowright A$ and $\beta : \Gamma \curvearrowright B$ are actions of countable, discrete amenable groups on unital, Kirchberg algebras then $\alpha \simeq \beta$ if and only if there exists a pointed isomorphism in $KK^{\Gamma}(\alpha, \beta)$.

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This also follows an existence–uniqueness–intertwining type strategy!

To make sense of this one needs the right notion of morphism between Γ-C*-algebras and a notion of Γ-unitary equivalence.

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Definition (Szabó 21')

Let Γ be a countable discrete group and $\alpha : \Gamma \curvearrowright A$, $\beta : \Gamma \curvearrowright B$ group actions. A cocycle morphism *from α to β consists of a *-homomorphism $\phi : A \to B$ and unitaries $\mathbb{u}_g \in U(M(B))$ for $g \in \Gamma$ s.t.*

1. $\phi\alpha_g = \mathrm{Ad}(\mathbb{u}_g)\beta_g\phi$
2. $\mathbb{u}_g\alpha_g(\mathbb{u}_h) = \mathbb{u}_{gh}$

$(\phi, \mathbb{u})$ is a cocycle conjugacy *if ϕ is an isomorphism, in which case we write $\alpha \simeq \beta$.*

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Definition (Szabó 21')

The category C^*_Γ has objects Γ-C*-algebas, morphisms cocycle morphisms with composition

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Proposition (Szabó 21')

The category C^*_Γ has a notion of unitary equivalence given by

$$(\mathrm{Ad}(u), u\alpha_g(u^*)) \in \mathrm{End}_{C^*_\Gamma}(\alpha) \quad u \in U(M(A))$$

Also C^*_Γ admits a complete metrisable toplogy at the level of morphisms yielding a notion of approximately unitary equivalence denoted $\approx_\Gamma$.

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Theorem (Szabó 21')

For Γ countable and discrete, the quotient functor $C^*_\Gamma \to C^*_\Gamma / \approx_\Gamma$ is full on isomorphisms.

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Actions of tensor categories

Let $(\mathcal{C}, \otimes)$ be a C*-tensor category e.g. $\mathrm{Rep}(\Gamma)$ or $\mathrm{Hilb}(\Gamma)$, we want to consider intertwining for $\mathcal{C}$-C*-algebras.

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We further assume that $\mathcal{C}$ is semisimple and has countably many isomorphism classes of simple objects.

We let $(\mathrm{Corr}(A), \boxtimes^{\mathrm{op}})$ be the C*-tensor category of non degenerate A-A-correspondences $\alpha : A \to \mathcal{L}(X_A)$ under the opposite internal tensor product.

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Definition

An action of $\mathcal{C}$ on A is a C-tensor functor*

$$(F, J) : (\mathcal{C}, \otimes) \to (\mathrm{Corr}(A), \boxtimes^{\mathrm{op}})$$

with $F(1_{\mathcal{C}}) = A$ where $J_{X,Y} : F(X) \boxtimes^{\mathrm{op}} F(Y) \cong F(X \otimes Y)$.

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e.g. if $\alpha : \Gamma \curvearrowright A$ is an action then

$$\mathrm{Hilb}(\Gamma) \to \mathrm{Corr}(A)$$
$$\mathbb{C}_g \mapsto {}_{\alpha_g}A$$

with the isomorphisms ${}_{\alpha_h}A \boxtimes {}_{\alpha_g}A \cong {}_{\alpha_g \alpha_h}A$ given by
$a \boxtimes b \mapsto \alpha_g(a)b$ is an action.

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Definition

A $\mathcal{C}$*-cocycle morphism* $(\phi, \mathbb{u}) : (F, J, A) \to (G, I, B)$ *consists of a* **-homomorphism* $\phi : A \to B$ *and coherent, natural, (possibly non-adjointable) bimodular isometries*

$$\mathbb{u}_X : F(X) \boxtimes {}_{\phi}B \to {}_{\phi}B \boxtimes G(X), \quad X \in \mathcal{C}.$$

With $\mathbb{u}_{1_{\mathcal{C}}}(a \boxtimes b) = \phi(a)b$. *It is moreover called a* *cocycle conjugacy* *if both* ϕ *is an isomorphism and* $\mathbb{u}_X$ *are unitaries.*

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With $\mathbb{u}_{1_\mathcal{C}}(a \boxtimes b) = \phi(a)b$.

Any non degenerate $\mathrm{Hilb}(\Gamma)$-cocycle morphism between group actions $\alpha : \Gamma \curvearrowright A$ and $\beta : \Gamma \curvearrowright B$ is given by a non degenerate *-homomorphism ϕ and bimodular isometries

$$\mathbb{u}_g : {}_{\phi\alpha_g} B \to {}_{\beta_g \phi} B$$

if we assume $\mathbb{u}_g$ is moreover a unitary then it is given by $b \mapsto \mathbb{v}_g^* b$ for some unitary $\mathbb{v}_g \in U(M(B))$ with $\phi\alpha_g = \mathrm{Ad}(\mathbb{v}_g)\beta_g\phi$. The cocycle identity for $\mathbb{v}_g$ holds by coherence.

Definition

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With $\mathbb{u}_{1_\mathcal{C}}(a \boxtimes b) = \phi(a)b$.

There is a much more user-friendly version. Note one can induce linear maps $F(X) \to G(X)$ from any cocycle morphism

$$F(X) \to F(X) \boxtimes {}_\phi B \xrightarrow{\mathbb{u}_X} {}_\phi B \boxtimes G(X) \to G(X)$$

Actions of tensor categories

Theorem (G., Neagu, see also Chen, Jones, Hernández Palomares)

Let $\phi : A \to B$ *be a* **-homomorphism, the data of admissible cocycle morphisms* $(\phi, \mathfrak{u}) : (F, J, A) \to (G, I, B)$ *is equivalent to a family of linear maps* $h^X : F(X) \to G(X)$ *such that*

- $h^X(a \triangleright \xi \triangleleft b) = \phi(a) \triangleright \xi \triangleleft \phi(b)$,
- $\langle h^X(\xi), h^X(\eta) \rangle = \phi(\langle \xi, \eta \rangle)$,
- $h^Y \circ F(f) = G(f) \circ h^X$, *for* $f \in \mathrm{Hom}(X, Y)$
- $I_{X,Y} \circ h^Y \boxtimes h^X = h^{X \otimes Y} \circ J_{X,Y}$,
- $h^{1_\mathcal{C}} = \phi$

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- $h^X(a \triangleright \xi \triangleleft b) = \phi(a) \triangleright \xi \triangleleft \phi(b)$, *bimodular*
- $\langle h^X(\xi), h^X(\eta) \rangle = \phi(\langle \xi, \eta \rangle)$, *isometric*
- $h^Y \circ F(f) = G(f) \circ h^X$, *natural*
- $I_{X,Y} \circ h^Y \boxtimes h^X = h^{X \otimes Y} \circ J_{X,Y}$, *coherent*
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A *cocycle morphism* *from* (F, J, A) *to* (G, I, B) *consists of a* **-homomorphism* $\phi : A \to B$ *and linear maps* h^X *for* $X \in \mathcal{C}$ *s.t.*

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- Straightforward composition $(\phi, h) \circ (\psi, l) = (\phi\psi, h \circ l)$.
- have a notion of inner automorphisms $\mathrm{Ad}(u)$ with linear maps $\mathrm{Ad}(u)_X(\xi) = u \triangleright \xi \triangleleft u^*$ for $u \in U(M(A))$, $\xi \in F(X)$.

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- have a notion of inner automorphisms $\mathrm{Ad}(u)$ with linear maps $\mathrm{Ad}(u)_X(\xi) = u \triangleright \xi \triangleleft u^*$ for $u \in U(M(A))$, $\xi \in F(X)$.
- the topology of the space of cocycle morphisms defined by the convergence

$$(\phi_\lambda, h_\lambda) \to (\phi, h) \Longleftrightarrow h_\lambda^X \to h^X \text{ pointwise}$$

is complete and metrisable.

Actions of tensor categories

Theorem (G, Neagu)

Let $\mathcal{C}$ be a semisimple C^*-tensor category with countably many isomorphism classes of simple objects. Then the quotient functor

$$C^*_{\mathcal{C}} \to C^*_{\mathcal{C}} / \approx_{\mathcal{C}}$$

is surjective on isomorphisms.

Outline

Equivariant $\mathcal{Z}$-stability

Classification Theorem

Unital, simple, separable, amenable, $\mathcal{Z}$-stable C*-algebra satisfying the UCT are classified by K-theory and traces.

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If A is $\mathcal{Z}$-stable and $\Gamma \curvearrowright A$ when is $A \rtimes \Gamma$ $\mathcal{Z}$-stable?

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"Nice" inclusions $A \subset B$ come from actions of unitary tensor categories and generalised crossed products.

Equivariant $\mathcal{Z}$-stability

F stabilised by $\mathcal{Z}$

$$\begin{aligned} F \otimes \mathrm{id}_{\mathcal{Z}} : \mathcal{C} &\to \mathrm{Corr}(A \otimes \mathcal{Z}) \\ X &\mapsto F(X) \otimes \mathcal{Z} \\ f &\mapsto F(f) \otimes \mathrm{id}_{\mathcal{Z}} \end{aligned}$$

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Definition

F is $\mathcal{Z}$-stable if $F \simeq F \otimes \mathrm{id}_{\mathcal{Z}}$.

Equivariant $\mathcal{Z}$-stability

Theorem (Kirchberg, see Toms–Winter)

Let A be a unital, separable C-algebra. Then*

$$A \otimes \mathcal{Z} \cong A \iff \mathcal{Z} \hookrightarrow A_\infty \cap A' \text{ unitaly.}$$

Equivariant $\mathcal{Z}$-stability

Theorem (Evington, G., Jones)

Let $F : \mathcal{C} \curvearrowright A$ be an action of a unitary tensor category with countably many isomorphism classes of simple objects on a unital, separable C-algebra. Then*

$$F \otimes \mathrm{id}_{\mathcal{Z}} \simeq F \iff \mathcal{Z} \hookrightarrow (A_\infty \cap A')^F \text{ unitaly}$$

where

$$(A_\infty \cap A')^F = \{a \in A_\infty : a \rhd \xi = \xi \lhd a,\ \forall X \in \mathcal{C}, \xi \in F(X)\}$$

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By hypothesis one can show that for any finite $F \subset A$, $G \subset A \otimes \mathcal{Z}$, $\varepsilon > 0$ there exists $v \in U(A \otimes \mathcal{Z})$ with

$$\| \mathrm{Ad}(v)(a \otimes 1) - a \otimes 1\| \underset{F,\varepsilon}{\approx} 0, \quad \mathrm{dist}(\mathrm{Ad}(v^*)G, A \otimes 1) < \epsilon$$

Equivariant $\mathcal{Z}$-stability

Sketch proof

By hypothesis one can show that for any finite $F \subset A$, $G \subset A \otimes \mathcal{Z}$, $\varepsilon > 0$ there exists $v \in U(A \otimes \mathcal{Z})$ with

$$\| \mathrm{Ad}(v)(a \otimes 1) - a \otimes 1 \| \underset{F,\varepsilon}{\approx} 0, \quad \mathrm{dist}(\mathrm{Ad}(v^*)G, A \otimes 1) < \epsilon$$

$$\begin{array}{ccccccc} A \otimes \mathcal{Z} & \longrightarrow & A \otimes \mathcal{Z} & \longrightarrow & A \otimes \mathcal{Z} \dots & \longrightarrow & A \otimes \mathcal{Z} \\ \uparrow{\scriptstyle \mathrm{id}_A \otimes 1} & & & & & & \\ A & \longrightarrow & A & \longrightarrow & A \dots & \longrightarrow & A \end{array}$$

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$$\begin{array}{ccccccc}
 & & G_1 & & & & \\
 & & \rotatebox{90}{$\subseteq$} & & & & \\
A \otimes \mathcal{Z} & \longrightarrow & A \otimes \mathcal{Z} & \longrightarrow & A \otimes \mathcal{Z} \ldots & \longrightarrow & A \otimes \mathcal{Z} \\
\uparrow {\scriptstyle \mathrm{id}_A \otimes 1} & \approx_{F_1,\epsilon_1} & \uparrow {\scriptstyle \operatorname{Ad}(v_1)(\mathrm{id}_A \otimes 1)} & & & & \\
A & \longrightarrow & A & \longrightarrow & A \ldots & \longrightarrow & A
\end{array}$$

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$$\begin{array}{ccccccc}
 & & G_1 & & G_2 & & \\
 & & \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & \\
A \otimes \mathcal{Z} & \longrightarrow & A \otimes \mathcal{Z} & \longrightarrow & A \otimes \mathcal{Z} \ldots & \longrightarrow & A \otimes \mathcal{Z} \\
{\scriptstyle \mathrm{id}_A \otimes 1} \uparrow & \approx_{F_1,\epsilon_1} & \uparrow & \approx_{F_2,\epsilon_2} & \uparrow {\scriptstyle \operatorname{Ad}(v_1 v_2)(\mathrm{id}_A \otimes 1)} & & \\
A & \longrightarrow & A & \longrightarrow & A \ldots & \longrightarrow & A
\end{array}$$

Sketch proof

One can show that for any finite $K \in \mathrm{Irr}(\mathcal{C})$, $F^X \subset F(X)$, $G^X \subset A \otimes \mathcal{Z}$, $\varepsilon > 0$ there exists $v \in U(A \otimes \mathcal{Z})$ s.t. for $X \in K$

$$\| \mathrm{Ad}(v)_X(\xi \otimes 1) - \xi \otimes 1 \| \underset{F^X,\varepsilon}{\approx} 0, \quad \mathrm{dist}(\mathrm{Ad}(v^*)_X G^X, F(X) \otimes 1) < \epsilon$$

$$\begin{array}{ccccccc}
 & & G_1^X & & G_2^X & & \\
 & & \cap\!\!| & & \cap\!\!| & & \\
F(X) \otimes \mathcal{Z} & \longrightarrow & F(X) \otimes \mathcal{Z} & \longrightarrow & F(X) \otimes \mathcal{Z} \ldots & \longrightarrow & F(X) \otimes \mathcal{Z} \\
\uparrow & \approx_{F_1^X,\epsilon_1} & \uparrow & \approx_{F_2^X,\epsilon_2} & \uparrow \, \mathrm{Ad}(v_1 v_2)(\mathrm{id}_{F(X)} \otimes 1) & & \\
F(X) & \longrightarrow & F(X) & \longrightarrow & F(X) \ldots & \longrightarrow & F(X)
\end{array} \quad \square$$

Equivariant $\mathcal{Z}$-stability

Theorem (Evington, G., Jones)

Let $F : \mathcal{C} \curvearrowright A$ be an action of a unitary tensor category with countably many isomorphism classes of simple objects on a unital, separable C^*-algebra then
F is $\mathcal{Z}$-stable $\Leftrightarrow \mathcal{Z} \hookrightarrow (A_\infty \cap A')^F$ unitaly

The right hand side is checkable in practice! We can check it for "stationary AF-actions" of fusion categories for example.

Thank you!